On the Wadge hierarchy of omega context free languages

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Abstract

The main result of this paper is that the length of the Wadge hierarchy of omega context free languages is greater than the Cantor ordinal ε_ω, which is the ω^{th} fixed point of the ordinal exponentiation of base ω. And the same result holds for the conciliating Wadge hierarchy, defined by J. Duparc (Wadge Hierarchy and Veblen Hierarchy: part 1: Borel Sets of Finite Rank, Journal of Symbolic Logic, 66 (2001), no. 1, 56–86), of infinitary context free languages, studied by D. Beauquier (Langages Algébriques Infinitaires, Ph. D. Thesis, Université Paris 7, 1984).

Key words: omega context free languages; infinitary context free languages; topological properties; Wadge hierarchy; conciliating Wadge hierarchy.

1 Introduction

In the sixties Büchi studied the ω-languages accepted by finite automata to prove the decidability of the monadic second order theory of one successor over the integers. Since then the so called ω-regular languages have been intensively studied, see [Tho90], [PP01] for many results and references. The extension to ω-languages accepted by pushdown automata has also been investigated, firstly by Cohen and Gold, Linna, Nivat, see S-taiger's paper [Sta97] for a survey of this work, including acceptation of infinite words by more powerful accepting devices, like Turing machines. A way to investigate the complexity of ω-languages is to consider their topological complexity. Mc Naugthon's Theorem implies that ω-regular languages are boolean combinations of Π^0_2-sets. We proved that omega context free languages (accepted by pushdown automata with a Büchi or Muller acceptance condition) exhaust the finite ranks of the Borel hierarchy, [Fin01a], that there exist some omega context free languages (ω-CFL) which are analytic but non Borel sets, [Fin00], and that there exist also some ω-CFL which are Borel sets of infinite rank. On the other side the Wadge Hierarchy of Borel sets is a great refinement of the Borel hierarchy and it induces on ω-regular languages the now called Wagner hierarchy which has been determined by Wagner in an effective way [Wag79]. Its length is the ordinal ω^ω. The Wadge Hierarchy of **deterministic** context free ω-languages has been recently determined [DFR01] [Dup99][Fin99b]. Its length is the ordinal $\omega^{(\omega^2)}$. We proved in [Fin99a] that the length of the Wadge hierarchy of context free ω-languages is an ordinal greater than or equal to the first fixed point of the ordinal exponentiation of base ω, the Cantor ordinal ε_0. We improve here this result and show that the length of the Wadge hierarchy of context free ω-languages is an ordinal greater than or equal to the ω^{th} fixed point of the ordinal exponentiation of base ω, the ordinal ε_ω. In order to get our results, we use recent results of Duparc. In [Dup01] [Dup95a] he gave a normal form of Borel $\Delta^{\mathbf{0}}_\omega$-sets, i.e. an inductive construction of a Borel set of every given degree in the Wadge hierarchy of $\Delta^{\mathbf{0}}_\omega$-Borel sets. In the course of the proof he studied the conciliating hierarchy which is a hierarchy of sets of finite **and** infinite sequences, closely connected to the Wadge hierarchy of non self dual sets. On the other hand the infinitary languages, i.e. languages containing finite **and** infinite words, accepted by pushdown automata have been studied in [Bea84a][Bea84b] where Beauquier considered these languages as process behaviours which may or may not terminate, as for transition systems studied in [AN82]. We study the conciliating hierarchy of infinitary context free languages and prove that the length of the conciliating hierarchy of infinitary context free languages is greater than the ordinal ε_ω.

2 ω-regular and ω-context free languages

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages,[Tho90], [Sta97]. We shall use usual notations of formal language theory. When Σ is a finite alphabet, a finite word over Σ is any sequence $x = x_1 \ldots x_k$, where $x_i \in \Sigma$ for $i = 1, \ldots , k$,and k is an integer ≥ 1. The length of x is k, denoted by $|x|$. If $|x| = 0$, x is the empty word denoted by λ. We write $x(i) = x_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$ and $x[0] = \lambda$. $\Sigma^\star$ is the set of finite words over Σ. The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where $a_i \in \Sigma, \forall i \geq 1$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$, where for all i $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$ is the finite word of length n, prefix of σ. The prefix relation is denoted $\sqsubseteq$: the finite word u is a prefix of the finite word v (respectively, the infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u.w$. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, the ω-power of V is the ω-language $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega / u_i \in V, \forall i \geq 1\}$. For $V \subseteq \Sigma^\star$, the complement of V (in $\Sigma^\star$) is $\Sigma^\star - V$ denoted V^-. For a subset $A \subseteq \Sigma^\omega$, the complement of A is $\Sigma^\omega - A$ denoted A^-. When we consider subsets of $\Sigma^{\leq\omega} = \Sigma^\star \cup \Sigma^\omega$, if $A \subseteq \Sigma^{\leq\omega}$ then $A^- = \Sigma^{\leq\omega} - A$. For any family L of finitary languages, the ω-Kleene closure of L, is: $\omega - KC(L) = \{\cup_{i=1}^n U_i.V_i^\omega / U_i, V_i \in L, \forall i \in [1, n]\}$.

Recall that the class REG_ω of ω-regular languages (or regular ω-languages) is the class of ω-languages accepted by finite automata with a Büchi or Muller acceptance condition. It is also the ω-Kleene closure of the class REG of regular finitary languages.
Similarly the class CFL_ω of ω-context free languages (ω-CFL) is the class of ω-languages accepted by pushdown automata with a Büchi or Muller acceptance condition. It is also the ω-Kleene closure of the class CFL of context free finitary languages, [Sta97].

If finite and infinite words are viewed as process behaviours, it is natural to consider the infinitary languages (containing finite **and** infinite words) recognized by transition systems [AN82]. The infinitary languages accepted by pushdown machines have been studied in [Bea84a], [Bea84b]. A pushdown machine is given with subsets K_1 and K_2 of its finite set of states K: K_1 is used for acceptation of finite words by final states (in K_1) and K_2 is used for acceptation of ω-words by a Büchi condition with the set K_2 as set of final states. The set of (finite or infinite) words accepted by the pushdown machine in such a way is the union of a finitary context free language and of an ω-CFL [Bea84a]. Then we let the following:

Definition 2.1 *Let X be a finite alphabet. A subset L of $X^{\leq\omega}$ is said to be an infinitary context free language iff there exists a finitary context free language $L_1 \subseteq X^\star$ and an ω-CFL $L_2 \subseteq X^\omega$ such that $L = L_1 \cup L_2$. The class of infinitary context free languages will be denoted $CFL_{\leq\omega}$.*

3 Borel and Wadge hierarchies

We assume the reader to be familiar with basic notions of topology which may be found in [LT94] [Mos80] and with the elementary theory of ordinals, including the operations of multiplication and exponentiation, which may be found in [Sie65]. Topology is an important tool for the study of ω-languages, and leads to characterization of several classes of ω-languages. For a finite alphabet X, we consider X^ω as a topological space with the Cantor topology. The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. The class of open sets of X^ω will be denoted by $\mathbf{G}$ or by $\Sigma^{\mathbf{0}}_{\mathbf{1}}$. The class of closed sets will be denoted by $\mathbf{F}$ or by $\Pi^{\mathbf{0}}_{\mathbf{1}}$. Define now the next classes of the Borel Hierarchy:

Definition 3.1 *The classes $\Sigma^{\mathbf{0}}_{\mathbf{n}}$ and $\Pi^{\mathbf{0}}_{\mathbf{n}}$ of the Borel Hierarchy on the topological space X^ω are defined as follows:*
$\Sigma^{\mathbf{0}}_{\mathbf{1}}$ is the class of open sets of X^ω.
$\Pi^{\mathbf{0}}_{\mathbf{1}}$ is the class of closed sets of X^ω.
$\Pi^{\mathbf{0}}_{\mathbf{2}}$ or $\mathbf{G}_\delta$ is the class of countable intersections of open sets of X^ω.
$\Sigma^{\mathbf{0}}_{\mathbf{2}}$ or $\mathbf{F}_\sigma$ is the class of countable unions of closed sets of X^ω.
And for any integer $n \geq 1$:
$\Sigma^{\mathbf{0}}_{\mathbf{n+1}}$ is the class of countable unions of $\Pi^{\mathbf{0}}_{\mathbf{n}}$-subsets of X^ω.
$\Pi^{\mathbf{0}}_{\mathbf{n+1}}$ is the class of countable intersections of $\Sigma^{\mathbf{0}}_{\mathbf{n}}$-subsets of X^ω.

The Borel Hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, for a countable ordinal α, are defined in the following way:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of X^ω in $\cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of X^ω in $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma$.

Recall some basic results about these classes, [Mos80]:

Proposition 3.2

(a) $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subsetneq \mathbf{\Sigma}^0_{\alpha+1} \cap \mathbf{\Pi}^0_{\alpha+1}$, for each countable ordinal $\alpha \geq 1$.
(b) $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma = \cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma \subsetneq \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$, for each countable limit ordinal α.
(c) A set $W \subseteq X^\omega$ is in the class $\mathbf{\Sigma}^0_\alpha$ iff its complement is in the class $\mathbf{\Pi}^0_\alpha$.
(d) $\mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha \neq \emptyset$ and $\mathbf{\Pi}^0_\alpha - \mathbf{\Sigma}^0_\alpha \neq \emptyset$ hold for every countable ordinal $\alpha \geq 1$.

We shall say that a subset of X^ω is a Borel set of rank α, for a countable ordinal α, iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

Introduce now the Wadge Hierarchy which is in fact a huge refinement of the Borel hierarchy:

Definition 3.3 *For $E \subseteq X^\omega$ and $F \subseteq Y^\omega$, E is said Wadge reducible to F ($E \leq_W F$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $E = f^{-1}(F)$.*
E and F are Wadge equivalent iff $E \leq_W F$ and $F \leq_W E$. This will be denoted by $E \equiv_W F$. And we shall say that $E <_W F$ iff $E \leq_W F$ but not $F \leq_W E$.
A set $E \subseteq X^\omega$ is said to be self dual iff $E \equiv_W E^-$, and otherwise it is said to be non self dual.

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The equivalence classes of $\equiv_W$ are called Wadge degrees.
WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$.
Remark that in the above definition, we consider that a subset $E \subseteq X^\omega$ is given together with the alphabet X.
We can now define the Wadge class of a set F:

Definition 3.4 *Let F be a subset of X^ω. The wadge class of F is $[F]$ defined by: $[F] = \{E/E \subseteq Y^\omega$ for a finite alphabet Y and $E \leq_W F\}$.*

Recall that each Borel class $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ is a Wadge class.
And that a set $F \subseteq X^\omega$ is a $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$)-complete set iff for any set $E \subseteq Y^\omega$, E is in $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$) iff $E \leq_W F$.

Theorem 3.5 (Wadge) *Up to the complement and $\equiv_W$, the class of Borel subsets of X^ω, for X a finite alphabet, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d^0_W from WH onto $|WH| - \{0\}$, such that for all $A, B \in WH$:*
$d^0_W A < d^0_W B \leftrightarrow A <_W B$ and

$d^0_W A = d^0_W B \leftrightarrow [A \equiv_W B \ or \ A \equiv_W B^-]$.

The Wadge hierarchy of Borel sets of **finite rank** has length ${}^1\varepsilon_0$ where ${}^1\varepsilon_0$ is the limit of the ordinals α_n defined by $\alpha_1 = \omega_1$ and $\alpha_{n+1} = \omega_1^{\alpha_n}$ for n a non negative integer, ω_1 being the first non countable ordinal. Then ${}^1\varepsilon_0$ is the first fixed point of the ordinal exponentiation of base ω_1. The length of the Wadge hierarchy of Borel sets in $\Delta^0_\omega = \Sigma^0_\omega \cap \Pi^0_\omega$ is the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1, which is a much larger ordinal. The length of the whole Wadge hierarchy of Borel sets is a huge ordinal. It is described in [Dup01] by the use of the Veblen functions.
There is an effective version of the Wadge hierarchy restricted to ω-regular languages:

Theorem 3.6 *For A and B some ω-regular sets, one can effectively decide whether $A \leq_W B$ and one can compute $d^0_W(A)$.*

The hierarchy obtained on ω-regular languages is now called the Wagner hierarchy and has length ω^ω. Wagner [Wag79] gave an automata structure characterization, based on notion of chain and superchain, for an automaton to be in a given class and then he got an algorithm to compute the Wadge degree of an ω-regular language. Wilke and Yoo proved in [WY95] that one can compute in polynomial time the Wadge degree of an ω-regular language. The Wagner hierarchy has an extension to deterministic context free as well as to deterministic Petri net ω-languages which has length $\omega^{(\omega^2)}$ [DFR01] [Dup99] [Fin99b].
The Wadge hierarchy restricted to ω-CFL is not effective: we have shown in [Fin01a] [Fin99a] [Fin00] that one can neither decide the Borel rank nor the Wadge degree of a Borel ω-CFL. In fact one cannot even decide whether an ω-CFL is a Borel set.

4 Operations on conciliating sets

4.1 Conciliating sets

We sometimes consider here subsets of $X^\star \cup X^\omega = X^{\leq\omega}$, for an alphabet X, which are called conciliating sets in [Dup01] [Dup95a]. In order to give a "normal form" of Borel sets in the Wadge hierarchy, J. Duparc studied the Conciliating hierarchy which is a hierarchy over conciliating sets closely related to the Wadge hierarchy. The two hierarchies are connected via the following correspondence:
First define A^d for $A \subseteq \Sigma^{\leq\omega}$ and d a letter not in Σ:

$$A^d = \{x \in (\Sigma \cup \{d\})^\omega \ / \ x(/d) \in A\}$$

where $x(/d)$ is the sequence obtained from x when removing every occurrence of the letter d. Then for $A \subseteq \Sigma^{\leq\omega}$ such that A^d is a Borel set, (we shall say in that case that A is a Borel conciliating set), A^d is always a non self dual subset of $(\Sigma \cup \{d\})^\omega$ and the correspondence $A \to A^d$ induces an isomorphism between the conciliating hierarchy and the Wadge hierarchy of non self dual sets. Hence we shall first concentrate on non self dual sets as in [Dup01] and we shall use the following definition of the Wadge degrees which is a slight modification of the previous one:

Definition 4.1 • $d_w(\emptyset) = d_w(\emptyset^-) = 1$
• $d_w(A) = sup\{d_w(B) + 1 \ / \ B$ non self dual and $B <_W A\}$
(for either A self dual or not, $A >_W \emptyset$).

Recall the definition of the conciliating degree of a conciliating set:

Definition 4.2 *Let $A \subseteq \Sigma^{\leq\omega}$ be a conciliating set over the alphabet Σ such that A^d is a Borel set. The conciliating degree of A is $d_c(A) = d_w(A^d)$.*

We recall now some properties of the correspondance $A \rightarrow A^d$ when context free languages are considered:

Proposition 4.3 ([Fin01a]) *a) if $A \subseteq \Sigma^\star$ is a context free (finitary) language, or if $A \subseteq \Sigma^\omega$ is an ω-CFL, then A^d is an ω-CFL.*
b) If A is the union of a finitary context free language and of an ω-CFL over the same alphabet Σ, then A^d is an ω-CFL over the alphabet $\Sigma \cup \{d\}$.

And we now introduce several operations over conciliating sets:

4.2 Operation of sum

Definition 4.4 ([Dup01]) *Assume that $X_A \subseteq X_B$ are two finite alphabets and that $X_B - X_A$ contains at least two elements and that $\{X_+, X_-\}$ is a partition of $X_B - X_A$ in two non empty sets. Let $A \subseteq X_A^{\leq\omega}$ and $B \subseteq X_B^{\leq\omega}$, then*

$$B + A = A \cup \{u.a.\beta \ / \ u \in X_A^\star, \ (a \in X_+ \text{ and } \beta \in B) \text{ or } (a \in X_- \text{ and } \beta \in B^-)\}$$

This operation is closely related to the ordinal sum as it is stated in the following:

Proposition 4.5 *Let $X_A \subseteq X_B$, $X_B - X_A$ containing at least two elements, and $A \subseteq X_A^{\leq\omega}$ and $B \subseteq X_B^{\leq\omega}$ such that A^d and B^d are Borel sets. Then $(B + A)^d$ is a Borel set and $d_c(B + A) = d_c(B) + d_c(A)$.*

Remark 4.6 *As indicated in Remark 5 of [Dup01], when $A \subseteq \Sigma^{\leq\omega}$ and X is a finite alphabet, it is easy to build $A' \subseteq (\Sigma \cup X)^{\leq\omega}$, such that $(A')^d \equiv_W A^d$. In fact A' can be defined as follows: for $\alpha \in (\Sigma \cup X)^{\leq\omega}$, let $\alpha \in A' \leftrightarrow \alpha' \in A$, where α' is α except each letter not in Σ is removed. Then in the sequel we assume that each alphabet is as enriched as desired, and in particular we can always define $B + A$ (or in fact another set C such that $C^d \equiv_W (B + A)^d$).*

Consider now conciliating sets which are union of a finitary CFL and of an ω-CFL.

Proposition 4.7 *Let $X_A \subseteq X_B$ such that $\{X_+, X_-\}$ is a partition of $X_B - X_A$ in two non empty sets. Assume $A \subseteq X_A^{\leq\omega}$ and $A, A^- \in CFL_{\leq\omega}$, and $B \subseteq X_B^{\leq\omega}$ and $B, B^- \in CFL_{\leq\omega}$. Then $B + A$ and $(B + A)^-$ are in $CFL_{\leq\omega}$.*

Definition 4.8 *Let $A \subseteq X_A^{\leq\omega}$ be a conciliating set over the alphabet X_A. Then $A.n$ is inductively defined by $A.1 = A$ and $A.(n+1) = (A.n) + A$, for each integer $n \geq 1$.*

4.3 Operation of exponentiation

Definition 4.9 *Let Σ be a finite alphabet and $\twoheadleftarrow \notin \Sigma$, let $X = \Sigma \cup \{\twoheadleftarrow\}$. Let x be a finite or infinite word over the alphabet $X = \Sigma \cup \{\twoheadleftarrow\}$.*
Then $x^{\twoheadleftarrow}$ is inductively defined by:
$\lambda^{\twoheadleftarrow} = \lambda$,
and for a finite word $u \in (\Sigma \cup \{\twoheadleftarrow\})^\star$:
$(u.a)^{\twoheadleftarrow} = u^{\twoheadleftarrow}.a$, *if $a \in \Sigma$,*
$(u. \twoheadleftarrow)^{\twoheadleftarrow} = u^{\twoheadleftarrow}$ *with its last letter removed if $|u^{\twoheadleftarrow}| > 0$,*
$(u. \twoheadleftarrow)^{\twoheadleftarrow} = \lambda$ *if $|u^{\twoheadleftarrow}| = 0$,*
and for u infinite:
$(u)^{\twoheadleftarrow} = \lim_{n\in\omega}(u[n])^{\twoheadleftarrow}$, *where, given β_n and v in $\Sigma^\star$,*
$v \sqsubseteq \lim_{n\in\omega} \beta_n \leftrightarrow \exists n \forall p \geq n \quad \beta_p[|v|] = v$.
(The finite or infinite word $\lim_{n\in\omega} \beta_n$ is determined by the set of its (finite) prefixes).

Remark 4.10 *For $x \in X^{\leq\omega}$, $x^{\twoheadleftarrow}$ denotes the string x, once every $^{\twoheadleftarrow}$ occuring in x has been "evaluated" to the back space operation (the one familiar to your computer!), proceeding from left to right inside x. In other words $x^{\twoheadleftarrow} = x$ from which every interval of the form "$a \twoheadleftarrow$" ($a \in \Sigma$) is removed.*

For example if $u = (a \twoheadleftarrow)^n$, for n an integer ≥ 1, or $u = (a \twoheadleftarrow)^\omega$, or $u = (a \twoheadleftarrow\twoheadleftarrow)^\omega$, then $(u)^{\twoheadleftarrow} = \lambda$. If $u = (ab \twoheadleftarrow)^\omega$ then $(u)^{\twoheadleftarrow} = a^\omega$ and if $u = bb(\twoheadleftarrow a)^\omega$ then $(u)^{\twoheadleftarrow} = b$.

We can now define the operation $A \to A^\sim$ of exponentiation of conciliating sets:

Definition 4.11 *For $A \subseteq \Sigma^{\leq\omega}$ and $\twoheadleftarrow \notin \Sigma$, let $X = \Sigma \cup \{\twoheadleftarrow\}$ and*
$A^\sim = \{x \in (\Sigma \cup \{\twoheadleftarrow\})^{\leq\omega} / x^{\twoheadleftarrow} \in A\}$.

The operation $\sim$ is monotone with regard to the Wadge ordering and produce some sets of higher complexity, as we shall see below. We now state that the operation of exponentiation of conciliating sets is closely related to ordinal exponentiation of base ω_1. We assume the reader to be familiar with the important notion of cofinality of a limit ordinal, which may be found in [Sie65] [CK73]. In the sequel we shall not have to consider cofinalities which are larger than ω_1.

Theorem 4.12 (Duparc [Dup01]) *Let $A \subseteq \Sigma^{\leq\omega}$ be a conciliating set such that A^d is a Borel set and $d_c(A) = d_w(A^d) = \alpha + n$ with α a limit ordinal and n an integer ≥ 0. Then $(A^\sim)^d$ is a Borel set and there are three cases:*

a) If $\alpha = 0$, then $d_c(A^\sim) = (\omega_1)^{d_c(A)-1}$
b) If α has cofinality ω, then $d_c(A^\sim) = (\omega_1)^{d_c(A)+1}$
c) If α has cofinality ω_1, then $d_c(A^\sim) = (\omega_1)^{d_c(A)}$

Consider now this operation $\sim$ over infinitary context free languages.

Theorem 4.13 ([Fin01a] [Fin99a]) *Whenever $A \subseteq \Sigma^\omega$ (respectively, $A \subseteq \Sigma^{\leq \omega}$) is in CFL_ω, (respectively, in $CFL_{\leq\omega}$), then $A^\sim$ is in CFL_ω, (respectively, in $CFL_{\leq\omega}$). And $A, A^- \in CFL_{\leq\omega}$ implies that $A^\sim, (A^\sim)^- = (A^-)^\sim \in CFL_{\leq\omega}$*

4.4 Operation of iterated exponentiation

One can already iterate the operation of exponentiation of sets. We shall use, in order to simplify our proofs, a variant $A^\approx$ of $A^\sim$ we already introduced in [Fin01a][Fin01b]. $A^\approx$ is defined as $A^\sim$ with the only difference that in the definition 4.9, we write: $(u. \twoheadleftarrow)^\twoheadleftarrow$ is undefined if $|u^\twoheadleftarrow| = 0$, instead of $(u. \twoheadleftarrow)^\twoheadleftarrow = \lambda$ if $|u^\twoheadleftarrow| = 0$. Then one can show that if $A \subseteq \Sigma^{\leq\omega}$ and $d_c(A) \geq 2$, then $A^\sim$ and $A^\approx$ are (conciliating) Wadge equivalent.
We define now, for a set $A \subseteq \Sigma^{\leq\omega}$: $A^{\approx.0} = A$, $A^{\approx.1} = A^\approx$ and $A^{\approx.(k+1)} = (A^{\approx.k})^\approx$, where we apply $k+1$ times the operation $A \to A^\approx$ with different new letters $\twoheadleftarrow_1, \twoheadleftarrow_2, \twoheadleftarrow_3, \ldots, \twoheadleftarrow_{k+1}$. But this way, from a Borel conciliating set of finite rank, we obtain only (conciliating) Borel sets of finite ranks, of Wadge degree $<^1 \varepsilon_0$. A way to get sets of higher degrees, is to define, for two letters a, b in Σ, the supremum of the sets $A^{\approx.i}$ by $\sup_{i\in\mathbb{N}} A^{\approx.i} = \bigcup_{i\in\mathbb{N}} a^i.b.A^{\approx.i}$. But this set is defined over an infinite alphabet, and any infinitary context free language is defined over a finite alphabet. So we have first to code this set over a finite alphabet. The conciliating set $A^{\approx.n}$ is defined over the alphabet $\Sigma \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n\}$ hence we have to code every eraser $\twoheadleftarrow_j$ by a finite word over a fixed finite alphabet. We shall code the eraser $\twoheadleftarrow_j$ by the finite word $\alpha.B^j.C^j.D^j.E^j.\beta$ over the alphabet $\{\alpha, B, C, D, E, \beta\}$. The reason of the coding we choose will be clear further, when we construct a pushdown automaton accepting an infinitary language close to the coding of $\sup_{i\in\mathbb{N}} A^{\approx.i}$, [Fin01c]. In fact this pushdown automaton needs to read four times the integer j characterizing the eraser $\twoheadleftarrow_j$.
Remark first that the morphism:
$F_n: \quad (\Sigma \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n\})^\star \to (\Sigma \cup \{\alpha, \beta, B, C, D, E\})^\star$
defined by $F(c) = c$ for each $c \in \Sigma$ and $F(\twoheadleftarrow_j) = \alpha.B^j.C^j.D^j.E^j.\beta$ for each integer $j \in [1, n]$, where $B, C, D, E, \alpha, \beta$ are new letters not in Σ, can be naturally extended to a function:
$\bar{F}_n: \quad (\Sigma \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_n\})^{\leq\omega} \to (\Sigma \cup \{\alpha, \beta, B, C, D, E\})^{\leq\omega}$.
We can now state the following lemma.

Lemma 4.14 *Let $A \subseteq \Sigma^{\leq\omega}$ such that $d_c(A) \geq 2$. Then $d_c(\bar{F}_n(A^{\approx.n})) = d_c(A^{\approx.n})$ holds for every integer $n \geq 1$, and $d_c(\sup_{i\in\mathbb{N}} \bar{F}_i(A^{\approx.i})) = \sup_{i\in\mathbb{N}} d_c(A^{\approx.i})$.*

But we can not show that, whenever $A \in CFL_{\leq\omega}$, then $\sup_{n\geq 1} \bar{F}_n(A^{\approx.n})$ is in $CFL_{\leq\omega}$. This is connected to the fact that the finitary language $\{B^jC^jD^jE^j \ /j \geq 1\}$ is not a context free language. But its complement is easily seen to be context free. Then we shall sligthly modify the set $\sup_{n\geq 1} \bar{F}_n(A^{\approx .n})$, in the following way. We can add to this language all $(\leq \omega)$-words in the form $a^n.b.u$ where there is in u a segment $\alpha.B^j.C^k.D^l.E^m.\beta$, with j,k,l,m integers ≥ 1, which does not code any eraser, or codes an eraser $\twoheadleftarrow_j$ for $j > n$.

Define first the following context free finitary languages over the alphabet
$X^\square = (\Sigma \cup \{\alpha, \beta, B, C, D, E\})$:
$L^B = \{a^n.b.u.B^j \ / \ n \geq 1 \text{ and } j > n \text{ and } u \in (X^\square)^\star\}$
$L^C = \{a^n.b.u.C^j \ / \ n \geq 1 \text{ and } j > n \text{ and } u \in (X^\square)^\star\}$
$L^D = \{a^n.b.u.D^j \ / \ n \geq 1 \text{ and } j > n \text{ and } u \in (X^\square)^\star\}$
$L^E = \{a^n.b.u.E^j \ / \ n \geq 1 \text{ and } j > n \text{ and } u \in (X^\square)^\star\}$
$L^{(B,C)} = \{u.\alpha.B^j.C^k.D^l.E^m.\beta \ / \ j,k,l,m \geq 1 \text{ and } j \neq k \text{ and } u \in (X^\square)^\star\}$
$L^{(C,D)} = \{u.\alpha.B^j.C^k.D^l.E^m.\beta \ / \ j,k,l,m \geq 1 \text{ and } k \neq l \text{ and } u \in (X^\square)^\star\}$
$L^{(D,E)} = \{u.\alpha.B^j.C^k.D^l.E^m.\beta \ / \ j,k,l,m \geq 1 \text{ and } l \neq m \text{ and } u \in (X^\square)^\star\}$

It is easy to show that each of these languages is a context free finitary language thus $L = L^B \cup L^C \cup L^D \cup L^E \cup L^{(B,C)} \cup L^{(C,D)} \cup L^{(D,E)}$ is also context free because the class CFL is closed under finite union. Then $L.(X^\square)^{\leq\omega}$ is an infinitary CFL. Remark that all words in $\sup_{n\geq 1} \bar{F}_n(A^{\approx .n})$ belong to the infinitary regular language $R = a^+.b.(\Sigma \cup (\alpha.B^+.C^+.D^+.E^+.\beta))^{\leq\omega}$. Consider now the language $L.(X^\square)^{\leq\omega} \cap R$. A word σ in this language is a word in R such that σ has an initial word in the form $a^n.b$, with $n \geq 1$, and σ contains a segment $\alpha.B^j.C^k.d^l.E^m.\beta$ with $j,k,l,m \geq 1$ which does not code any eraser $\twoheadleftarrow_j$ or codes such an eraser but with $j > n$. Define now

$$A^\bullet = \sup_{n\geq 1} \bar{F}_n(A^{\approx .n}) \cup [L.(X^\square)^{\leq\omega} \cap R]$$

Introduce now some notations for ordinals. For an ordinal α we define $\omega_1(1,\alpha) = \omega_1^\alpha$ and for an integer $n \geq 1$, $\omega_1(n+1,\alpha) = \omega_1^{\omega_1(n,\alpha)}$. If $\alpha \leq {}^1\varepsilon_0$ the limit of the ordinals $\omega_1(n,\alpha)$ is the ordinal ${}^1\varepsilon_0$. And if $\alpha > {}^1\varepsilon_0$ the limit of the sequence of ordinals $\omega_1(n,\alpha)$ is the first fixed point of the operation of exponentiation of base ω_1 which is greater than (or equal to) α. We shall denote it ${}^1\varepsilon_0(\alpha)$. Then one can enumerate the sequence of the ω first fixed points of the operation $\alpha \to \omega_1^\alpha$, which are: ${}^1\varepsilon_0$, ${}^1\varepsilon_1 = {}^1\varepsilon_0({}^1\varepsilon_0 + 1)$, ${}^1\varepsilon_2 = {}^1\varepsilon_0({}^1\varepsilon_1 + 1)$, and for each integer $n \geq 0$: ${}^1\varepsilon_{n+1} = {}^1\varepsilon_0({}^1\varepsilon_n + 1)$. The next fixed point is the ω^{th} fixed point, denoted ${}^1\varepsilon_\omega$, and it is also the limit of the sequence of fixed points ${}^1\varepsilon_n$, for $n \geq 0$: ${}^1\varepsilon_\omega = \sup_{n\in\omega}({}^1\varepsilon_n)$. The sequence of fixed points of the operation of exponentiation of base ω_1 continues beyond this ordinal (because, for each ordinal α, there exists such a fixed point which is greater than α), but we shall not need larger ordinals. We can now state the following:

Theorem 4.15 *Let $A \subseteq \Sigma^{\leq\omega}$ be an infinitary context free language such that $d_c(A) \geq 2$. Then $A^\bullet$ is an infinitary context free language such that $d_c(A^\bullet)$ is the ordinal ${}^1\varepsilon_0(d_c(A))$.*

Remark that in particular if $2 \leq d_c(A) <^1 \varepsilon_0$, i.e. if A^d is Borel of finite rank and of Wadge degree ≥ 2, then $d_c(A^\bullet) =^1 \varepsilon_0$, and $A^\bullet$ is a Borel set of rank ω.
The proof of $[A \in CFL_{\leq\omega} \to A^\bullet \in CFL_{\leq\omega}]$ relies on a technical construction of a

pushdown automaton accepting $A^\bullet$ from a pushdown automaton accepting A. The idea of the construction is already in [Fin01b] where we proved the existence of some ω-CFL which are Borel sets of infinite rank.

5 Wadge hierarchy of infinitary context free languages

If we consider the operation of ordinal exponentiation of base ω: $\alpha \to \omega^\alpha$, one can define in a similar way as above the successive fixed points of this operation. These ordinals are the well known Cantor ordinals ε_0, ε_1, ... and ε_ω is the ω^{th} such fixed point, [Sie65]. From the preceding closure properties of the class $CFL_{\leq\omega}$ under the operations of sum, of exponentiation and of iterated exponentiation, and using the correspondence between these operations and the arithmetical operations over ordinals, one can show the following:

Theorem 5.1 *The length of the conciliating hierarchy of infinitary languages in $CFL_{\leq\omega}$ is greater than ε_ω. The length of the Wadge hierarchy of context free ω-languages is greater than ε_ω.*

The proof will be detailed in the full version of this paper. Remark that in fact the length of the Wadge hierarchy of ω-CFL being Borel sets of rank ω is still greater than ε_ω. We see again that, considering their topological complexity, non deterministic pushdown automata have a much stronger expressive power than deterministic pushdown automata, when reading ω-words with a Büchi or Muller acceptance condition. Next it remains to determine the exact length of the Wadge hierarchy of Borel ω-CFL and all the degrees of ω-CFL.

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